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Imaginary number

An **imaginary number** is the product of a real number and the imaginary unit i,[note 1] which is defined by its property $i^2 = -1$.[1][2] The square of an imaginary number bi is $-b^2$. For example, $5i$ is an imaginary number, and its square is -25. The number zero is considered to be both real and imaginary.[3]

The powers of i are cyclic:
$\vdots$
$i^{-2} = -1$
$i^{-1} = -i$
$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
$\vdots$
i is a 4th root of unity

Originally coined in the 17th century by René Descartes[4] as a derogatory term and regarded as fictitious or useless, the concept gained wide acceptance following the work of Leonhard Euler in the 18th century, and Augustin-Louis Cauchy and Carl Friedrich Gauss in the early 19th century.

An imaginary number bi can be added to a real number a to form a complex number of the form $a + bi$, where the real numbers a and b are called, respectively, the *real part* and the *imaginary part* of the complex number.[5]

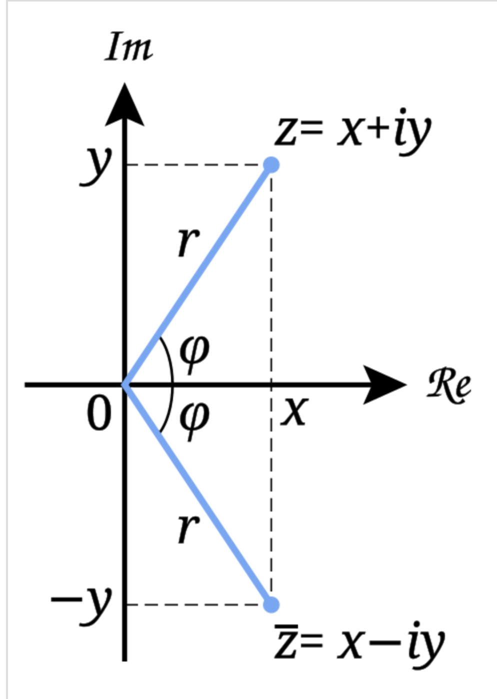
History

Although the Greek mathematician and engineer Heron of Alexandria is noted as the first to present a calculation involving the square root of a negative number,[6][7] it was Rafael Bombelli who first set down the rules for multiplication of complex numbers in 1572. The concept had appeared in print earlier, such as in work by Gerolamo Cardano. At the time, imaginary numbers and negative numbers were poorly understood and were regarded by some as fictitious or useless, much as zero once was. Many other mathematicians were slow to adopt the use of imaginary numbers, including René Descartes, who wrote about them in his *La Géométrie* in which he coined the term *imaginary* and meant it to be derogatory.[8][9] The use of imaginary numbers was not widely accepted until the work of Leonhard Euler (1707–1783) and Carl Friedrich Gauss (1777–1855). The geometric significance of complex numbers as points in a plane was first described by Caspar Wessel (1745–1818).[10]

In 1843, William Rowan Hamilton extended the idea of an axis of imaginary numbers in the plane to a four-dimensional space of quaternion imaginaries in which three of the dimensions are analogous to the imaginary numbers in the complex field.

Geometric interpretation

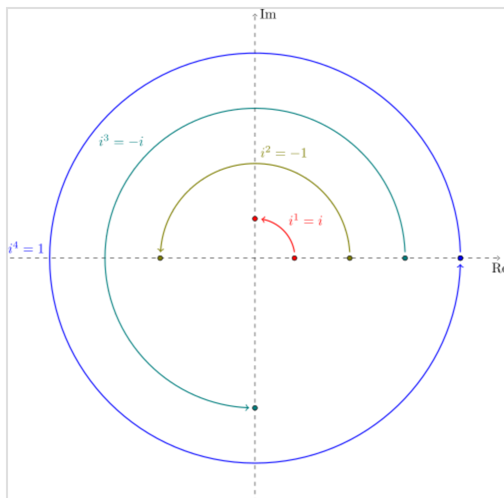
Geometrically, imaginary numbers are found on the vertical axis of the complex number plane, which allows them to be presented perpendicular to the real axis. One way of viewing imaginary numbers is to consider a standard number line positively increasing in magnitude to the right and negatively increasing in magnitude to the left. At 0 on the x-axis, a y-axis can be drawn with "positive" direction going up; "positive" imaginary numbers then increase in magnitude upwards, and "negative" imaginary numbers increase in magnitude downwards. This vertical axis is often called the "imaginary axis"[11] and is denoted $i\mathbb{R}$, $\mathbb{I}$, or $\Im$.[12]



An illustration of the complex plane. The imaginary numbers are on the vertical coordinate axis.

In this representation, multiplication by i corresponds to a counterclockwise rotation of 90 degrees about the origin, which is a quarter of a circle. Multiplication by $-i$ corresponds to a clockwise rotation of 90 degrees about the origin. Similarly, multiplying by a purely imaginary number bi, with b a real number, both causes a counterclockwise rotation about the origin by 90 degrees and scales the answer by a factor of b. When $b < 0$, this can instead be described as a clockwise rotation by 90 degrees and a scaling by $|b|$.[13]

Square roots of negative numbers



90-degree rotations in the complex plane

Care must be used when working with imaginary numbers that are expressed as the principal values of the square roots of negative numbers.[14] For example, if x and y are both positive real numbers, the following chain of equalities appears reasonable at first glance:

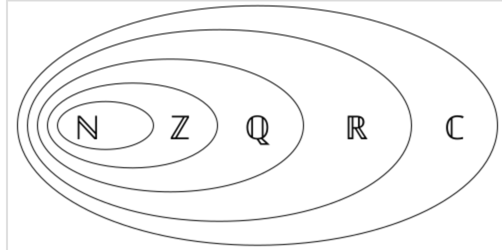
$$\sqrt{x \cdot y} = \sqrt{(-x) \cdot (-y)} \overset{\text{(fallacy)}}{=} \sqrt{-x} \cdot \sqrt{-y} = i\sqrt{x} \cdot i\sqrt{y} = -\sqrt{x \cdot y}.$$

But the result is clearly nonsense. The step where the square root was broken apart was illegitimate. (See Mathematical fallacy.)

See also

- −1

- Dual number
- Split-complex number



Set inclusions between the natural numbers (ℕ), the integers (ℤ), the rational numbers (ℚ), the real numbers (ℝ), and the complex numbers (ℂ)

Notes

1. j is usually used in engineering contexts where i has other meanings (such as electrical current)

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8. Descartes, René, *Discours de la méthode* (Leiden, (Netherlands): Jan Maire, 1637), appended book: *La Géométrie*, book three, p. 380. From page 380: (https://gallica.bnf.fr/ark:/12148/btv1b86069594/f464.item.zoom) *"Au reste tant les vrayes racines que les fausses ne sont pas tousjours reelles; mais quelquefois seulement imaginaires; c'est a dire qu'on peut bien tousjours en imaginer autant que jay dit en chasque Equation; mais qu'il n'y a quelquefois aucune quantité, qui corresponde a celles qu'on imagine, comme encore qu'on en puisse imaginer trois en celle cy,* x^3 *– 6xx + 13x – 10 = 0, il n'y en a toutefois qu'une reelle, qui est 2, & pour les deux autres, quoy qu'on les augmente, ou diminue, ou multiplie en la façon que je viens d'expliquer, on ne sçauroit les rendre autres qu'imaginaires."* (Moreover, the true roots as well as the false [roots] are not always real; but sometimes only imaginary [quantities]; that is to say, one can always imagine as many of them in each equation as I said; but there is sometimes no quantity that corresponds to what one imagines, just as although one can imagine three of them in this [equation], x^3 – 6xx + 13x – 10 = 0, only one of them however is real, which is 2, and regarding the other two, although one increase, or decrease, or multiply them in the manner that I just explained, one would not be able to make them other than imaginary [quantities].)
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- Nahin, Paul (1998). *An Imaginary Tale: the Story of the Square Root of −1* (https://archive.org/details/imaginarytales00nahi). Princeton: Princeton University Press. ISBN 0-691-02795-1., explains many applications of imaginary expressions.

External links

- How can one show that imaginary numbers really do exist? (https://www.math.toronto.edu/mathnet/answers/imagexist.html) – an article that discusses the existence of imaginary numbers.
- 5Numbers programme 4 (https://www.bbc.co.uk/radio4/science/5numbers4.shtml) – BBC Radio 4 programme
- Why Use Imaginary Numbers? (http://www2.dsu.nodak.edu/users/mberg/Imaginary/imaginary.htm) Archived (https://web.archive.org/web/20190825172656/http://www2.dsu.nodak.edu/users/mbe

rg/Imaginary/imaginary.htm) 2019-08-25 at the Wayback Machine – Basic Explanation and Uses of Imaginary Numbers

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